Int. J. Heat Mass Transfer. Vol. 23, pp. 655–662
Pergamon Press Ltd. 1980. Printed in Great Britain

REFINED CLOSED METHODS FOR THE CONTRA-FLOW THERMAL REGENERATOR PROBLEM

A. J. WILLMOTT
Department of Computer Science, University of York, Heslington, York, YO1 5DD, U.K.
and
R. C. DUGGAN
Department of Mathematics, University of Exeter, Exeter, U.K.

(*Received* 4 *October* 1979 *and in revised form* 6 *November* 1979)

Abstract—Earlier workers have presented closed form partly analytical solutions to the problem of the contra-flow thermal regenerator. These methods of regenerator calculation have not proved to be as robust as perhaps was anticipated when the methods were first devised. In this paper are described proposals whereby some of these difficulties can be alleviated. The proposals are relevant to the possible development of the closed methods for realistic non-linear models of regenerators.

NOMENCLATURE

a_j, coefficient of jth term in (21);
A, heating surface areas [m^2];
C, matrix specific heat [J/kg K];
$F(\xi)$, matrix temperature distribution at the start of a period [K];
h, heat transfer coefficient [W/m^2 K];
J_1, Bessel function of the first kind and first order;
K, function defined by (19);
L, length of regenerator [m];
m, mass of gas resident in regenerator [kg];
M, mass of solid matrix [kg];
n, order of the power series in (21);
R, denotes matrix in (22);
P, period [s];
S, gas specific heat [J/kg K];
t, gas temperature [K];
T, solid matrix temperature [K];
W, mass flow rate of gas [kg/s];
y, distance down regenerator from current entrance [m].

Greek symbols

ξ, dimensionless distance from current entrance, defined in (5);
η, dimensionless time in current period, defined in (6);
Λ, reduced length defined in (7);
Π, reduced period defined in (8);
Θ, time [s];
η_{REG}, thermal ratio defined in (23).

Superscripts

$'$, refers to hot period;
$''$, refers to cold period.

INTRODUCTION

IN 1961, Nahavandi and Weinstein [1] presented a method of solution of the differential equations

$$\frac{\partial}{\partial \xi} t(\xi,\eta) = T(\xi,\eta) - t(\xi,\eta), \tag{1}$$

$$\frac{\partial}{\partial \eta} T(\xi,\eta) = t(\xi,\eta) - T(\xi,\eta) \tag{2}$$

describing the temperature behaviour of the regenerative heat exchanger. This procedure was identical to that given by Nusselt [2] in 1927. Whereas Nusselt presented his solution which he acknowledged to be due to Riemann, Nahavandi and Weinstein produced the same integral equation solution to the partial differential equations (1) and (2) using the method of Laplace transforms.

By solving the integral equations by a numerical method, Nahavandi and Weinstein calculated the solid temperature distributions at the end of each period of operation at cyclic equilibrium. These temperature distributions were represented as power series in the dimensionless distance variable ξ. The coefficients of the power series were calculated by solving a set of simultaneous linear algebraic equations.

The successful application of the Nahavandi and Weinstein method relies upon the absence of ill-conditioning of the resulting linear algebraic equations. Should any ill-conditioning arise, the method will break down. The objectives of this paper are two-fold. Firstly, the method proposed by Nahavandi and Weinstein for the solution of the integral equations is discussed and the circumstances under which the method breaks down are explained. Secondly a possible means of alleviating these difficulties is discussed.

It is important to note that both the Nahavandi and Weinstein and the Iliffe [3] methods are closed in the sense that the cyclic equilibrium condition of the regenerator, that is when the regenerator performance has become periodic, is computed directly. In the open methods, the regenerator model is, possibly, run through many cycles from an arbitrary starting condition until cyclic equilibrium is achieved. It might seem therefore that closed methods are ideal when the regenerator under consideration is particularly sluggish, and many cycles must be computed in an open method before equilibrium is achieved. This will be particularly the case where non-linear models of regenerator are being developed in which are embodied, for example, temperature dependent thermal properties of both gas and packing (chequerwork), time varying gas flow rates and strongly temperature dependent radiative heat transfer effects. The considerations here for the idealised regenerator are equally applicable to possible developments of closed methods for such non-linear models.

In this paper, therefore, the possibly unexpected way in which closed methods break down is described in the hope that computer programs developed for realistic non-linear models will at least be designed to be able to anticipate these difficulties for program users not familiar with the mathematical methods involved.

THE IDEALISED REGENERATOR

The regenerator model incorporating the differential equations (1) and (2) embodies certain idealisations. In this linear model the hot gas enters with constant temperature at one end of the heat storing matrix, loses some of its heat as it passes through the matrix before it departs with a time varying temperature at the cold end. The hot gas is then shut off and it is assumed that all the residual hot gas is driven from the channels of the matrix. In his paper of 1929 Hausen [4] supposed the idealised regenerator to be equipped with special pistons to effect the evacuation of this residual gas. At this stage the cold gas enters the regenerator at the cold end with constant temperature. The heat stored in the matrix is regenerated by the cold gas which leaves at the hot end with a variable higher temperature. Again, the residual gas is ejected from the matrix channels before another cycle of operations begins.

The cycle of operations is said to consist of a 'hot period' followed by a 'cold period', and after a sufficiently large number of such cycles, the thermal behaviour of the regenerator becomes periodic and 'cyclic equilibrium' is said to have been established. It is important to note that such an equilibrium is characteristic of a forced oscillation. Although from a practical view point, the regenerator is considered to promote the exchange of heat between the gases, this aspect is exploited in the mathematical analysis that the alternate passage of the hot and cold gas through the regenerator channels imposes periodic variations of temperature in the heat storing material.

In the Nahavandi and Weinstein paper, a solution to the differential equations is obtained directly for this cyclic equilibrium condition.

The physical idealisation involves a number of further simplifying assumptions:

(a) the thermal conductivity of the gases and the matrix is zero in a direction parallel to that of the gas stream. The temperature variations within the matrix in the radial direction are not considered. It is assumed that the thermal conductivity of the matrix in the radial direction is either infinite, in which case the solid will be isothermal in this radial direction, or to be finite. In the latter case, a bulk heat-transfer coefficient is developed which incorporates the surface resistance to heat transfer and the resistance internal to the heat storing solid. The use of this bulk heat-transfer coefficient is discussed in detail by Willmott [5]. Its use does not affect the essential form of the differential equations (1) and (2);

(b) the heat-transfer coefficients and the thermal properties of the gas and solid are regarded as temperature independent;

(c) the mass flow rate of the gas does not vary with time in each period, although the flow rate in the hot period may be different from that in the cold period.

MATHEMATICAL REPRESENTATION OF THE IDEAL REGENERATOR

The descriptive differential equations

$$hA(t - T) = MC\frac{\partial T}{\partial \Theta}, \tag{3}$$

$$hA(T - t) = WSL\frac{\partial t}{\partial y} + mS\frac{\partial t}{\partial \Theta} \tag{4}$$

have been derived in the Nahavandi and Weinstein paper which also introduces the dimensionless parameters ξ and η. The parameters take the form

$$\xi = \frac{hA}{WSL}y \tag{5}$$

$$\eta = \frac{hA}{MC}\left(\Theta - \frac{m}{WL}y\right). \tag{6}$$

By introducing these parameters, equations (3) and (4) assume the form of equations (1) and (2) introduced earlier in this paper. In these equations one or two primes are inserted against the various symbols when the hot or the cold period is considered respectively.

Corresponding to each period of the cycle are the dimensionless parameters 'reduced length', Λ, and 'reduced period', Π, originally proposed by Hausen [4],

$$\Lambda = \frac{hA}{WS} \tag{7}$$

$$\Pi = \frac{hA}{MC}(P - m/W). \tag{8}$$

(Nahavandi and Weinstein use ξ_0 for Λ', ξ_0^* for Λ'', η_0 for Π' and η_0^* for Π''.)

The differential equations (1) and (2) are solved subject to two boundary conditions, namely:

(a) The inlet gas temperature during each period does not vary with time. Inspection of the linearity of equations (1) and (2) reveals that the temperature scale is immaterial and in the analysis, the hot inlet temperature is considered to be 1 while the cold inlet temperature is set equal to 0.

(b) The 'reversal condition' incorporates the facts that distance is always measured from the gas entrance in both periods, and that the temperature at any position in the heating storing matrix at the end of one period is equal to that at the same position at the commencement of the next period. The same position measured to be ξ' from the gas entrance in hot period is measured to be ξ'' from the gas entrance in cold period and, in incorporating the contra-flow mode of operation, ξ' is related to ξ'' by $\xi'/\Lambda' = 1 - \xi''/\Delta''$.

The reversal boundary conditions take the form:

$$T'(\xi',\Pi') = T''(\Lambda''\{1 - \xi'/\Lambda'\},0) \tag{9}$$

$$T''(\xi'',\Pi'') = T'(\Lambda'\{1 - \xi''/\Lambda''\},0). \tag{10}$$

SYMMETRIC REGENERATORS

The thermal regenerator problem is simplified by consideration of the symmetric case where $\Lambda = \Lambda' = \Lambda''$ and $\Pi = \Pi' = \Pi''$. Here the temperature performance of the solid in the hot period is exactly symmetric to that in the cold period at cyclic equilibrium. In these particular circumstances, the reversal condition can be written

$$T'(\xi',0) = 1 - T''(\xi'',0) \tag{11}$$

using the (0, 1) temperature scale. This means that the problem is reduced to the 'single period' boundary value problem. The reversal condition can then be written

$$T'(\xi',0) + T'(\Lambda - \xi',\Pi) = 1. \tag{12}$$

The difficulties which are associated with both the method of Nahavandi and Weinstein and that of Iliffe manifest themselves in both the symmetric and more general cases. Discussion of these problems is considerably simplified, without significant loss of generality, by investigation of this symmetric case.

THE INTEGRAL EQUATIONS

It is useful to introduce a simplifying notation at this stage: it is specified that the solid matrix temperature distribution at the start of the hot period is $F'(\xi')$ and at the start of the cold period is $F''(\xi'')$. Thus

$$F'(\xi') = T'(\xi',0) \tag{13}$$

$$F''(\xi'') = T''(\xi'',0). \tag{14}$$

Nusselt and subsequently Nahavandi and Weinstein represented the solution of the differential equations (1) and (2) in the form:

$$T''(\xi'',\eta'') = e^{-\eta''}F''(\xi'') + \int_0^{\xi} \frac{iJ_1\{2i\sqrt{[(\xi''-\varepsilon)\eta'']}\}}{\sqrt{[(\xi''-\varepsilon)\eta'']}} \times \eta''\, e^{-(\eta''+\xi''-\varepsilon)}F''(\varepsilon)d\varepsilon \tag{15}$$

for the cold period and

$$T'(\xi',\eta') = 1 - e^{-\eta'}[1 - F'(\xi')] + \int_0^{\xi} \frac{iJ_1\{2i\sqrt{[(\xi'-\varepsilon)\eta']}\}}{\sqrt{[(\xi'-\varepsilon)\eta']}} \times \eta'\, e^{-(\eta'+\xi'-\varepsilon)}(1 - F'(\varepsilon))d\varepsilon \tag{16}$$

for the hot period.

Upon application of the reversal conditions (9) and (10) and setting $\eta'' = \Pi''$ in equation (15) and $\eta' = \Pi'$ in equation (16), the solid temperature distributions $F'(\xi')$ and $F''(\xi'')$ at the start of the hot period and the start of the cold period respectively are related by the integral equations

$$F'[\Lambda'(1 - \xi''/\Lambda'')] = e^{-\Pi''}F''(\xi'') + \int_0^{\xi''} K''(\xi''-\varepsilon)F''(\varepsilon)d\varepsilon \tag{17}$$

and

$$1 - F''(\Lambda''(1 - \xi'/\Lambda')) = e^{-\Pi'}[1 - F'(\xi')] + \int_0^{\xi'} K'(\xi'-\varepsilon)[1 - F'(\varepsilon)]d\varepsilon \tag{18}$$

where

$$K(\xi-\varepsilon) = \frac{-iJ_1\{2i\sqrt{[(\xi-\varepsilon)\Pi]}\}}{\sqrt{[(\xi-\varepsilon)\Pi]}}\Pi\, e^{-(\Pi+\xi-\varepsilon)}. \tag{19}$$

For the symmetric case, application of the reversal condition (12) for the cold period yields the integral equation

$$F''(\Lambda - \xi) + e^{-\Pi}F''(\xi) + \int_0^{\xi} K(\xi-\varepsilon)F''(\varepsilon)d\varepsilon = 1. \tag{20}$$

SOLUTION OF THE INTEGRAL EQUATION FOR THE SYMMETRIC CASE

In the Nahavandi and Weinstein approach, the temperature distribution $F(\xi)$ [omitting now the double primes in equation (20)] is represented by an approximating polynomial

$$F(\xi) = \sum_{j=0}^{n} a_j\xi^j.$$

Equation (20) now takes the following form:

$$\sum_{j=0}^{n} a_j(\Lambda-\xi)^j + e^{-\Pi}\sum_{j=0}^{n} a_j\xi^j + \sum_{j=0}^{n} a_j\int_0^{\xi} \varepsilon^j K(\xi-\varepsilon)d\varepsilon = 1. \tag{21}$$

By applying this equation (21) at $n+1$ distinct values $\xi_i (i = 0, 1, 2, \ldots, n)$ of ξ on the range $0 \leqslant \xi \leqslant \Lambda$, a set of $n+1$ simultaneous linear equations is obtained in the $n+1$ unknown coefficients $a_0, a_1, a_2, \ldots, a_n$. These equations are written in the form

$$R\mathbf{a} = \mathbf{1} \tag{22}$$

where $\mathbf{a}$ is the column vector $\{a_0, a_1, a_2, \ldots, a_n\}$ and $\mathbf{1}$ is the column vector $\{1, 1, 1, \ldots, 1\}$. R is a $(n+1) \times (n+1)$ matrix, the (i, j)th element of which is

$$(\Lambda - \xi_i)^j + e^{-\Pi}\xi_i^j + \int_0^{\xi_i} \varepsilon^j K(\xi_i - \varepsilon)\mathrm{d}\varepsilon$$

for $0 \leqslant i, j \leqslant n$.

Computer program libraries commonly provide subroutines for the evaluation of the first order Bessel function J_1 with imaginary arguments. However, it is worth noting that it can be demonstrated that

$$\lim_{x \to 0} K(x) = \Pi e^{-\Pi}$$

and that this limiting value can be used for $x < 10^{-4}$ with negligible relative error for $\Pi < 5$. Many practical values of Π are in the range $0.1 < \Pi < 3$.

In order to evaluate the elements of the matrix R, the integration

$$\int_0^{\xi_i} \varepsilon^j K(\xi_i - \varepsilon)\mathrm{d}\varepsilon$$

can be achieved using Gaussian quadrature.

CALCULATION OF REGENERATOR EFFECTIVENESS

The effectiveness of regenerator thermal behaviour is measured in terms of the thermal ratio η_{REG} which is defined as the ratio of the actual heat transfer rate in a contra-flow exchanger of infinite heat transfer area. This is represented by the equation

$$\eta_{REG} = \frac{\Lambda''}{\Pi''}\left[\frac{1}{\Lambda''}\int_0^{\Lambda''} F''(\xi'')\mathrm{d}\xi'' - \frac{1}{\Lambda'}\int_0^{\Lambda'} F'(\xi')\mathrm{d}\xi'\right] \tag{23}$$

which for the symmetric regenerator takes the form

$$\eta_{REG} = \frac{1}{\Pi}\int_0^{\Lambda} (F''(\xi) - F'(\xi))\mathrm{d}\xi. \tag{24}$$

Applying the reversal condition (12), the expression (24) for the thermal ratio takes the form

$$\eta_{REG} = \frac{1}{\Pi}\int_0^{\Lambda} (F''(\xi) + F''(\Lambda - \xi) - 1)\mathrm{d}\xi. \tag{25}$$

Upon substituting the polynomial expansion, the integral

$$\eta_{REG} = \frac{1}{\Pi}\int_0^{\Lambda}\left\{\sum_{j=0}^{n} [a_j(\xi^j + (\Lambda - \xi)^j)] - 1\right\}\mathrm{d}\xi \tag{26}$$

can be evaluated explicitly using

$$\eta_{REG} = \frac{2}{\Pi}\left(\sum_{j=0}^{n} \frac{a_j \Lambda^{j+1}}{j+1} - \Lambda\right). \tag{27}$$

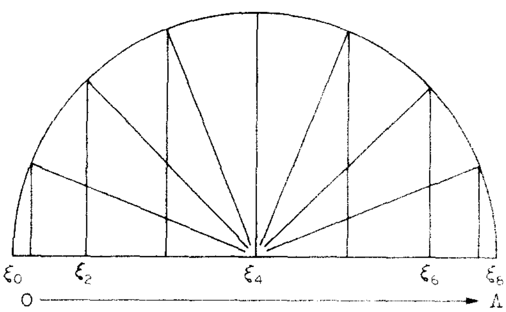

FIG. 1. Chebyshev distribution of data points.

Thus upon the solution of the simultaneous linear equations (22) for the coefficients $\{a_0, a_1, a_2, \ldots, a_n\}$, the thermal ratio for a symmetric regenerator of reduced length Λ and reduced period Π can be calculated using equation (27).

DIFFICULTIES IN SOLVING THE LINEAR ALGEBRAIC EQUATIONS

The successful application of the methods of regenerator calculation described here depends upon the absence of ill-conditioning of the equations (22). Although the Nahavandi and Weinstein method does not suffer from ill-conditioning as such, another problem does arise. For increasing values of reduced length, for a fixed value of reduced period Π, the determinant $|R|$ from equations (22) increases in value. What happens is that certain elements of the matrix R become very large relative to the remainder, and as a consequence the linear equations become difficult to solve.

It will be recalled that the (i, j)th element of the matrix R is

$$(\Lambda - \xi_i)^j + e^{-\Pi}\xi_i^j + \int_0^{\xi} \varepsilon^j K(\xi_i - \varepsilon)\mathrm{d}\varepsilon$$

for $0 \leqslant i, j \leqslant n$. Representation of the temperature distribution $F''(\xi)$ by a polynomial of degree 8 for example in a regenerator of reduced length $\Lambda = 10$ involved elements of the matrix of order at least 10^8 in size. The smaller the value of ξ_i, the larger will be the corresponding matrix element. Of course, the larger the degree of the polynomial required for accurate solution, the more severe will be this problem. As the linear equations (22) are solved in the Nahavandi and Weinstein method, division of an element by another very large matrix element generates a very small element; this gives the false impression that the equations are ill-conditioned, and a computer simulation of the regenerator may be halted in this unexpected manner.

ITERATIVE CALCULATION OF REGENERATOR EFFECTIVENESS

Using the Nahavandi and Weinstein method of calculation, the thermal ratio of a regenerator of specified reduced length Λ and reduced period Π is first computed assuming $F''(\xi)$ to be a linear function of ξ. The computation is repeated using successively higher powers in the polynomial expansion of ξ, representing $F''(\xi)$. This iterative process is said to have 'converged' when the difference between two consecutive evaluations of η_{REG} is less than some prescribed level, ε. In the results presented in Tables 1(a), 1(b), and 1(c), the value of ε was set at 10^{-5}.

CHOICE OF DATA POINTS ξ_i

Nahavandi and Weinstein chose to solve the set of simultaneous linear equations (22) for values $\xi_i = i\Lambda/n$, for $i = 0, 1, 2, \ldots, n$ which are equally spaced apart.

In the middle of a regenerator, the temperature of the solid varies linearly with ξ and η in both periods. Non-linear temperature behaviour is propagated from the entrances to the regenerator by the constant inlet gas temperatures in both periods. This would suggest that the data points ξ_i should be more closely clustered in some way around the entrances than in the middle of the regenerator. From a mathematical point of view, it is known that if it is required to interpolate a polynomial $Q(x)$ of degree n by another polynomial $P(x)$ of degree $n-1$, the maximum value of error $|Q(x) - P(x)|$ is minimised if the interpolation points are chosen to be the Chebyshev points.

For the solution of the equations (22), such Chebyshev points ξ_i were used where

$$\xi_i = \frac{\Lambda}{2}\left\{1 - \cos\frac{i\pi}{n}\right\}, \quad i = 0, 1, 2, \ldots$$

and the results are displayed in Tables 1(a), and 1(c). These Chebyshev points are clustered around the regenerator entrances in a manner illustrated for $n = 8$ in Fig. 1. Note, however, that for $n = 1$ and $n = 2$ the Chebyshev points and the equally spaced points coincide. It will be observed that a reduction by one or at most two in the degree of the power expansion required for convergence in the calculation of the thermal ratio is achieved by selecting the Chebyshev points. In one sense, this is only a marginal gain. Nevertheless the method of evaluation is not complicated by this facility and advantage should be taken of this modification. In another sense this gain is important. Most elements of the matrix R require lengthy calculations including the numerical evaluation of the definite integrals

$$\int_0^{\xi_i} \xi^j K(\xi_i - \xi)\mathrm{d}\varepsilon$$

and, indeed, this time always exceeds the time taken to solve the simultaneous linear equations. While this may not be an important consideration where the regenerator calculations are undertaken on a large, fast computing facility, it will be important if the calculation is performed on a small desk top computer, as is becoming the fashion in some laboratories. However, the most important gain is that the likelihood of the method breaking down is minimised if, by the use of the Chebyshev data points, the degree of the polynomial required is held as low as possible.

CONCLUDING REMARKS

Willmott and Kulakowski [6] mentioned the problem of the possible instability of closed methods for regenerator simulations. They suggested a numerical acceleration scheme which might be applied to open methods of thermal regenerator calculations, and thereby completely bypassed any problems that might be associated with closed methods. Previously, Willmott and Thomas [7] had concluded that the Iliffe [3] closed method should not be applied to 'long' regenerators, where $\Lambda/\Pi > 3$ and $\Lambda > 10$.

However, the possibility of using Gaussian quadrature methods to achieve high accuracy in closed methods of the Iliffe type were discussed by Hausen [8]. Reference should be made to the original paper as to the scheme proposed there. Whether this Hausen scheme could help to overcome the instability problems discussed by Willmott and Thomas remains to be determined.

In this paper we have shown that regenerator calculation methods of the Nahavandi and Weinstein closed type can be readily refined in such a way that the calculation time is reduced and the range of the applicability of such methods is extended. That the refined method described here generates the correct solution to the differential equations (1) and (2) can be readily checked by comparing the solutions presented in Tables 1(a), 1(b) and 1(c) with those obtained by different methods and presented elsewhere by Hausen [4, 9], Iliffe [3] and Willmott [5]. The essence of the refinement lies in the choice of the Chebyshev data points, rather than equally spaced data points. There is little doubt that the use of such Chebyshev points should be considered in any development of closed methods for the regenerator non-linear models mentioned earlier in the paper. For a method in which the solid temperature distribution at the start of a period (hot/cold) at cyclic equilibrium is approximated by a power series, as in the Nahavandi and Weinstein method, no additional computation effort is involved in the calculation of the regenerator effectiveness. Thus any future developments of closed methods for thermal regenerator simulations which embody, for example, temperature dependent physical properties of the heat storing mass or the fluids, or time varying flow rates, should endeavour to exploit the advantages of using the Chebyshev data points, and if possible to develop the Nahavandi and Weinstein approach for such non-linear problems.

Acknowledgements—The authors acknowledge the significant contribution of Nahavandi and Weinstein to regenerator theory in their original paper upon which the developments described here are founded. We wish to thank the University of York Computing Service for running the programs developed by one of us, R. C. Duggan, on the ICL 4130 computer. In addition we must thank Mr. W. J. Marwood of Gilling, Yorkshire, formerly of the National Physical Laboratory at Teddington, who prepared the diagram. However, most important, the authors wish to thank Professor H. Hausen of the Technical University of Hannover for indirectly prompting us to publish this paper. The paper in

Table 1(a). Thermally symmetrical regenerators with $\Pi = 1$. Values of η_{REG} against η, degree of power expansion for (a) equally spaced points (b) Chebyshev points

Λ		$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$
1	(a)	0.3210	0.3221	0.3221					
	(b)		0.3221						
2	(a)	0.4871	0.4909	0.4911	0.4912	0.4912			
	(b)			0.4912	0.4912				
3	(a)	0.5867	0.5921	0.5936	0.5937	0.5937			
	(b)			0.5938	0.5937	0.5937			
4	(a)	0.6529	0.6600	0.6619	0.6621	0.6622	0.6622		
	(b)			0.6622	0.6622				
5	(a)	0.7004	0.7075	0.7104	0.7108	0.7109	0.7109		
	(b)			0.7110	0.7110	0.7109	0.7109		
6	(a)	0.7364	0.7428	0.7465	0.7472	0.7474	0.7474		
	(b)			0.7475	0.7475	0.7474	0.7474		
7	(a)	0.7647	0.7701	0.7744	0.7753	0.7757	0.7757		
	(b)			0.7757	0.7759	0.7758	0.7757	0.7757	
8	(a)	0.7875	0.7919	0.7966	0.7977	0.7982	0.7983	0.7983	0.7984
	(b)			0.7981	0.7986	0.7984	0.7984		
9	(a)	0.8064	0.8098	0.8146	0.8158	0.8166	0.8167	0.8168	0.8168
	(b)			0.8164	0.8171	0.8169	0.8168	0.8168	
10	(a)	0.8222	0.8249	0.8295	0.8309	0.8319	0.8321	0.8322	
	(b)			0.8315	0.8325	0.8322	0.8322		

Table 1(b). Thermally symmetrical regenerators with $\Pi = 2$. Values of η_{REG} against η, degree of power expansion for (a) equally spaced points; (b) Chebyshev points

Λ		$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$
1	(a)	0.2904	0.2930	0.2930					
	(b)			0.2930					
2	(a)	0.4557	0.4660	0.4664	0.4665	0.4665			
	(b)			0.4665	0.4665				
3	(a)	0.5556	0.5740	0.5755	0.5757	0.5757			
	(b)			0.5758	0.5757	0.5757			
4	(a)	0.6213	0.6450	0.6486	0.6490	0.6491	0.6491		
	(b)			0.6492	0.6491	0.6491			
5	(a)	0.6682	0.6941	0.7003	0.7010	0.7012	0.7012		
	(b)			0.7015	0.7012	0.7012			
6	(a)	0.7039	0.7294	0.7384	0.7396	0.7399	0.7400	0.7400	
	(b)			0.7403	0.7401	0.7400	0.7400		
7	(a)	0.7324	0.7559	0.7675	0.7693	0.7698	0.7698	0.7699	0.7699
	(b)			0.7702	0.7701	0.7699	0.7699		
8	(a)	0.7560	0.7766	0.7902	0.7926	0.7935	0.7936	0.7936	
	(b)			0.7938	0.7940	0.7936	0.7936		
9	(a)	0.7760	0.7933	0.8084	0.8114	0.8127	0.8129	0.8129	
	(b)			0.8128	0.8135	0.8129	0.8129		
10	(a)	0.7931	0.8073	0.8233	0.8268	0.8285	0.8288	0.8289	0.8289
	(b)			0.8284	0.8297	0.8290	0.8289	0.8289	

Table 1(c). Thermally symmetrical regenerators with $\Pi = 3$. Values of η_{REG} against η, degree of power expansion for (a) equally spaced points; (b) Chebyshev points.

Λ		$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
1	(a)	0.2530	0.2560	0.2560			
	(b)			0.2560			
2	(a)	0.4162	0.4303	0.4305	0.4305		
	(b)			0.4305	0.4305		
3	(a)	0.5184	0.5466	0.5477	0.5477		
	(b)			0.5477	0.5477		
4	(a)	0.5849	0.6250	0.6280	0.6282	0.6282	
	(b)			0.6283	0.6282	0.6282	
5	(a)	0.6312	0.6790	0.6850	0.6856	0.6856	
	(b)			0.6859	0.6856	0.6856	
6	(a)	0.6660	0.7169	0.7269	0.7280	0.7280	
	(b)			0.7285	0.7280	0.7280	
7	(a)	0.6938	0.7443	0.7586	0.7603	0.7605	0.7605
	(b)			0.7613	0.7606	0.7605	0.7605
8	(a)	0.7171	0.7646	0.7832	0.7857	0.7861	0.7861
	(b)			0.7870	0.7863	0.7861	0.7861
9	(a)	0.7373	0.7800	0.8027	0.8060	0.8068	0.8068
	(b)			0.8077	0.8072	0.8067	0.8067
10	(a)	0.7551	0.7924	0.8183	0.8225	0.8238	0.8238
	(b)			0.8246	0.8244	0.8238	0.8238

an earlier form was written 5 yr ago and sent to Hausen for his comments. Subsequently Hausen outlined the contents in his book [9], but was only able to refer to the paper as a private communication from the authors.

REFERENCES

1. A. H. Nahavandi and A. S. Weinstein, A solution to the periodic flow regenerative heat exchanger problem, *Appl. Scient. Res.* **10**, 335–348 (1961).
2. W. Nusselt, Die Theorie des Winderhitzers, *Z. Ver. Dt. Ing.* **71**, 85–91 (1927).
3. C. E. Iliffe, Thermal analysis of the contra-flow regenerative heat exchanger, *Proc. Instn. Mech. Engrs.* **159**, 363–371 (1948).
4. H. Hausen, Über die Theorie des Wärmeaustausches in Regeneratoren, *Z. Angew. Math. Mech.* **9**, 173–200 (1929).
5. A. J. Willmott, The regenerative heat exchanger computer representation, *Int. J. Heat Mass Transfer* **12**, 997–1014 (1969).
6. A. J. Willmott and B. T. Kulakowski, Numerical acceleration of thermal regenerator simulations, *Int. J. Num. Methods Eng.* **11**, 533–551 (1977).
7. A. J. Willmott and R. J. Thomas, Analysis of the long contra-flow regenerative heat exchanger, *J. Inst. Maths Applics.* **14**, 267–280 (1974).
8. H. Hausen, Berechnumg von Regeneratoren nach der Gaußschen Intergrationsmethode, *Int. J. Heat Mass Transfer* **17**, 1111–1113 (1974).
9. H. Hausen, Wärmeübertragung im Gegenstrom, Gleichstrom und Kreuzstrom, 2nd edition, Springer-Verlag, Berlin (1976).

METHODES AFFINES POUR LES PROBLEMES DES REGENERATEURS THERMIQUES A CONTRE-COURANT

Résumé—Des chercheurs ont présenté des solutions analytiques du problème du régénérateur thermique à contre-courant. Ces méthods de calcul n'ont pas été prouvées aussi robustes qu'on le souhaitait à leur mise en place. Dans cet article, on décrit des aménagements où quelques unes de ces difficultés peuvent être réduites. Les propositions permettent le développement de méthodes adaptées aux modèles non linéaires des régénérateurs.

VERFEINERTE GESCHLOSSENE LÖSUNGSMETHODEN FÜR DAS PROBLEM DES GEGENSTROM-REGENERATORS

Zusammenfassung—Frühere Autoren haben für das Problem des Gegenstrom-Regenerators teilanalytische Lösungen in geschlossener Form vorgelegt. Es hat sich herausgestellt, daß diese Berechnungsmethoden nicht so unempfindlich sind, wie vielleicht angenommen wurde, als man die Verfahren erstmals angewandt hat. In dieser Arbeit werden Vorschläge beschrieben, durch die einige dieser Schwierigkeiten umgangen werden können. Die Vorschläge beziehen sich auf eine mögliche Weiterentwicklung der geschlossenen Verfahren für realistische nicht-lineare Regenerator-Modelle.

УСОВЕРШЕНСТВОВАНИЕ ТОЧНЫХ МЕТОДОВ РАСЧЕТА РЕГЕНЕРАТИВНОГО ПРОТИВОТОЧНОГО ТЕПЛООБМЕННИКА

Аннотация — Ранее некоторыми исследователями были предложены точные полуаналитические решения задач противотока в регенеративном теплообменнике. Однако эти методы оказались не столь надежными, как предполагалось. В данной работе описаны способы, посредством которых некоторые из недостатков этих методов могут быть устранены. Эти предложения позволяют развить точные методы решения для реальных нелинейных моделей регенеративных теплообменников.